Problem 1 (28 points). Decide if each of the following claims is true or false. Justify your answer with a brief argument (for example a reference to the textbook, including exercises done in class, a short proof, or a counterexample).

(i) $C[0,1]$ is a Banach space when equipped with the norm $\|f\|_2 = \left(\int_0^1 |f(x)|^2\,dx\right)^{1/2}$.

(ii) $C[0,1]$ is a Banach space when equipped with the norm $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$.

(iii) If M is a closed subspace of a Hilbert space H and if $M^\perp = \{0\}$, then $M = H$.

(iv) If H is a Hilbert space with a complete orthonormal set $(e_n)_{n=1}^\infty$, then $\sum_{n=1}^\infty \frac{1}{n} e_n$ defines an element in H.

(v) If $f \in C[-\pi,\pi]$ has Fourier coefficients $c_n(f)$, $n \in \mathbb{Z}$, then $\sum_{n=-\infty}^\infty |c_n(f)|^2 < \infty$.

(vi) Every linear operator $T\colon E \to F$ between two normed spaces E and F is bounded.

(vii) Every continuous linear operator $T\colon E \to F$ between two normed spaces E and F is bounded.

Problem 2 (18 points). Consider the vector space $C[0,1]$ equipped with the inner product

$$(f,g) = \int_0^1 f(x)\overline{g(x)}\,dx, \qquad f,g \in C[0,1],$$

and associated norm $\|f\| = (f,f)^{1/2}$. Let M be the subspace of $C[0,1]$ spanned by the functions $u_1(x) = 1$ and $u_2(x) = x$. Let $h \in C[0,1]$ be given by $h(x) = \sqrt{x}$.

(i) Find an orthonormal basis for M.

(ii) Find $a, b \in \mathbb{C}$ which minimize the norm $\|h - au_1 - bu_2\|$. Sketch the graphs of h and $au_1 + bu_2$ in the same coordinate system.

(iii) Determine $\|h\|$, $\|h - au_1 - bu_2\|$, and $\|au_1 + bu_2\|$, where a, b are as in question (ii).

Problem 3 (16 points). Consider the Hilbert space $L^2(-\pi,\pi)$ with its usual inner product

$$(f,g) = \int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx, \qquad f,g \in L^2(-\pi,\pi).$$

Let $(e_n)_{n=-\infty}^\infty$ be the orthonormal basis for $L^2(-\pi,\pi)$ consisting of the trigonometric functions

$$e_n(x) = \frac{1}{\sqrt{2\pi}}\,e^{inx}, \qquad x \in [-\pi,\pi],$$