and let $c_n(f) = (f, e_n)$ denote the Fourier coefficients of $f \in L^2(-\pi, \pi)$.

Consider the functions $f, g\colon [-\pi, \pi] \to \mathbb{C}$ given by $f(x) = |x|$ and $g(x) = \sin(x)$. You may use that $\sin(x) = (e^{ix} - e^{-ix})/2i$, and you are not asked to prove this fact.

(i) Calculate $\|f\|^2$ and $\|g\|^2$.

(ii) Determine the Fourier coefficients $c_n(f)$ and $c_n(g)$ for all $n \in \mathbb{Z}$.

(iii) Determine $\sum_{n=-\infty}^{\infty} c_n(f)\overline{c_n(g)}$.

Problem 4 (8 points). Consider the Hilbert space $L^2(-1, 1)$ and the linear functional F on $L^2(-1, 1)$ given by

$$F(f) = \int_{-1}^{1} t f(t)\, dt, \qquad f \in L^2(-1, 1).$$

Show that F is bounded and determine $\|F\|$. Find $f \in L^2(-1, 1)$ such that $\|f\| = 1$ and $|F(f)| = \|F\|$.

Problem 5 (20 points). Consider the Hilbert space ℓ^2 and consider the linear operator $S\colon \ell^2 \to \ell^2$ given by

$$S(x_1, x_2, x_3, x_4, x_5, x_6, \dots) = (x_2, x_1, x_4, x_3, x_6, x_5, \dots), \qquad x = (x_1, x_2, \dots) \in \ell^2,$$

i.e.,

$$Sx = y = (y_n)_{n=1}^{\infty}, \qquad y_n = \begin{cases} x_{n+1}, & n \text{ odd} \\ x_{n-1}, & n \text{ even.} \end{cases}$$

You are given, and you need not prove, the facts that S is linear and that $Sx \in \ell^2$ for all $x \in \ell^2$.

(i) Show that $S \in \mathscr{L}(\ell^2)$ and determine $\|S\|$.

(ii) Show that $S^* = S$ and that $S^2 = I$.

(iii) Put $P = \frac{1}{2}(I + S)$. Show that P is a projection, i.e., show that $P = P^*$ and that $P = P^2$. [Hint: You may use, without giving a proof, that I is Hermitian, i.e., $I^* = I$. You may also use that the identity operator I acts as a multiplicative unit on $\mathscr{L}(\ell^2)$, i.e., $IT = TI = T$ for all $T \in \mathscr{L}(\ell^2)$.]

(iv) Put

$$x = (1, 2, 3, 4, 3, 2, 1, 0, 0, 0, \dots) \in \ell^2.$$

Determine the vectors Px and $x - Px$ and calculate the norms $\|x\|$, $\|Px\|$, and $\|x - Px\|$.